

AN APPLICATION OF RATIONAL HYPERPLANE ARRANGEMENTS IN COUNTING INDEPENDENT SETS OF GRAPHS

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ABSTRACT. We develop a method in counting independent sets of disjoint union of certain types of graphs. This method is based upon the n to 1 covering from the points in the finite field \mathbb{F}_q^n which the characteristic polynomial of a rational hyperplane arrangement in \mathbb{R}^n are counting to the independent sets of the corresponding graph. We show that for graphs $G(k)$ with k vertices corresponding to a class of rational hyperplane arrangements, the number of n -element independent sets of any disjoint union $G = G(k_1) + G(k_2) + \cdots + G(k_s)$ depends only on the total number of vertices in the entire disjoint union, $\sum_i k_i$. We also give some results with broader conditions. This new technique has importance in simplifying the complexity of counting independent sets in a disjoint union of graphs, and provides closed-form solutions for certain cases.

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1. INTRODUCTION

An independent set of a graph is a subset of its vertices, none two of which are connected by an edge. Counting independent sets of graphs is in fact a prototypical $\#P$ -complete problem [1] in graph theory. For example, Figure 1 illustrates one of 11-element independent sets in a disjoint union of three graphs. Therefore, we would like to find out what kind of graph structures makes counting and enumeration problems solvable.

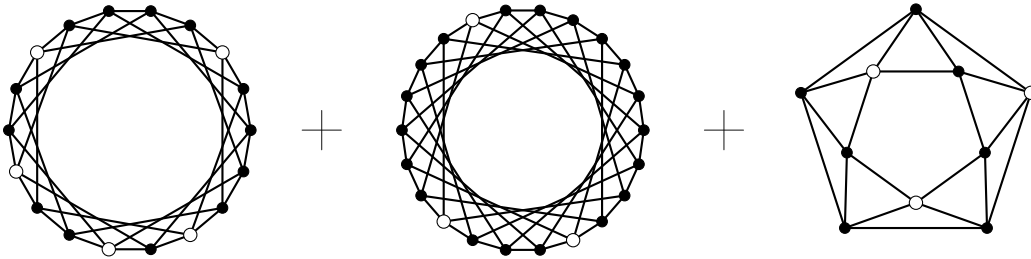


FIGURE 1. One 11-element independent set of a disjoint union of three graphs. Each open-circle vertex is an element of this independent set in the disjoint union.

The theory of hyperplane arrangements has been vitally developed over the past decades [2–5]. The motivation of this work is to extend applications of hyperplane arrangements to independence problems of disjoint unions of certain types of graphs.

A (real) hyperplane arrangement is a finite collection \mathcal{A} of hyperplanes in \mathbb{R}^n . For each hyperplane arrangement \mathcal{A} , we associate its intersection poset (partially ordered set), $L(\mathcal{A})$, which consists of all intersections of subsets of the hyperplanes in \mathcal{A} ordered by reverse inclusion. $L(\mathcal{A})$ has a unique minimal element $\hat{0} = \mathbb{R}^n$. We call \mathcal{A} central if $\cap_{H \in \mathcal{A}} H \neq \emptyset$. Denote by $\text{rank}(\mathcal{A})$ the rank of \mathcal{A} which is the dimension of the space spanned by the normals to the hyperplanes in \mathcal{A} . The characteristic polynomial $\chi_{\mathcal{A}}(t)$ is defined as $\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) t^{\dim(x)}$, where $\mu(\hat{0}, x)$ is the Möbius function of $L(\mathcal{A})$ [2, Lecture 1]. The characteristic polynomial can also be simplified through:

Theorem 1.1. (*Whitney’s Theorem*) [2, Thm 2.4] *Let \mathcal{A} be an arrangement in \mathbb{R}^n . Then,*

$$\chi_{\mathcal{A}_n}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}, \mathcal{B} \text{ central}} (-1)^{\#\mathcal{B}} t^{n - \text{rank}(\mathcal{B})}.$$

We call an arrangement \mathcal{A} essential if $\text{rank}(\mathcal{A}) = n$. Denote by $r(\mathcal{A})$ the number of regions which are the connected components of $\mathbb{R}^n - \cup_{H \in \mathcal{A}} H$ and denote $b(\mathcal{A})$ to be the number of relatively bounded regions [2, Lecture 1], which exactly means bounded when \mathcal{A} is essential. We have the following theorem to algebraically calculate $r(\mathcal{A})$ and $b(\mathcal{A})$.

Theorem 1.2. (*Zaslavsky’s Theorem*) [5] *For any hyperplane arrangement \mathcal{A} in \mathbb{R}^n , we have*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1), \quad b(\mathcal{A}) = (-1)^{\text{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1).$$

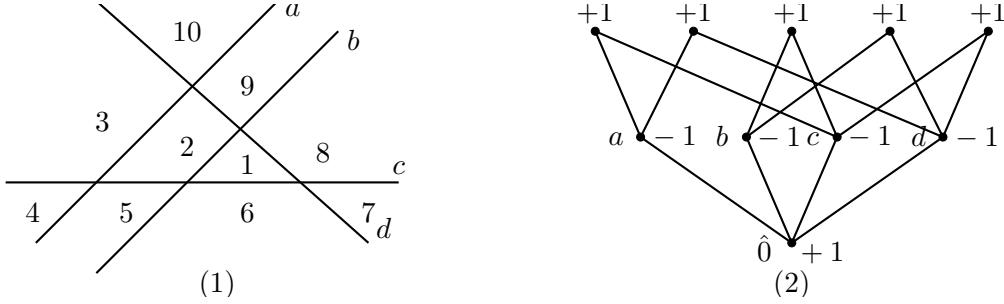


FIGURE 2. A hyperplane arrangement $\{a, b, c, d\}$ (1) and its intersection poset (2).

Figure 2 illustrates an example of an essential hyperplane arrangement in \mathbb{R}^2 and its corresponding intersection poset, yielding $\chi_{\mathcal{A}}(t) = t^2 - 4t + 5$. Note that $r(\mathcal{A}) = 10$ and $b(\mathcal{A}) = 2$, i.e., among a total of 10 regions, two regions are bounded, as shown Figure 2(1).

For calculating characteristic polynomial of rational hyperplane arrangements, whose hyperplanes are defined over integers, we have the following important finite field method.

Theorem 1.3. (*Finite Field Theorem*) [6, Thm 2.2] *Let \mathcal{A} be any rational hyperplane arrangement in \mathbb{R}^n and q be a large enough prime number. Then*

$$\chi_{\mathcal{A}}(q) = \#(\mathbb{F}_q^n - \cup_{H \in \mathcal{A}} H) = q^n - \# \cup_{H \in \mathcal{A}} H.$$

The Catalan arrangement $\mathcal{C}_n = \{x_i - x_j = 0, 1, -1, 1 \leq i < j \leq n\}$ has a lot of connections to our later constructions. Its characteristic polynomial can be computed through the finite field method:

$$\chi_{\mathcal{C}_n}(t) = t(t - n - 1)(t - n - 2) \cdots (t - 2n + 1).$$

To relate graph theory and hyperplane arrangements, the most basic construction is: for a simple graph G on $[n] = \{1, \dots, n\}$, let \mathcal{A}_G be the arrangement in \mathbb{R}^n with hyperplanes $x_i - x_j = 0$ for

every $ij \in E(G)$, which is called the graphical arrangement. In fact, every graphical arrangement is a subarrangement of Braid arrangement $\mathcal{B}_n = \{x_i = x_j \mid 1 \leq i < j \leq n\}$, which is the graphical arrangement of the complete graph.

We generalize the above finite field result to

$$\mathcal{A}_n = \mathcal{B}_n \cup \{x_i = a_k x_j + b_k \mid 1 \leq i \neq j \leq n, 1 \leq k \leq m\}$$

for a_i, b_i 's being non-negative integers. In the following proposition, we establish a connection between the arrangement and a corresponding graph by a different construction.

Proposition 1.4. *Let $\mathcal{A}_n = \mathcal{B}_n \cup \{x_i = a_k x_j + b_k \mid 1 \leq i \neq j \leq n, 1 \leq k \leq m\}$ for some $a_k, b_k \in \mathbb{N}$, $(a_k, b_k) \neq (1, 0)$, be a rational hyperplane arrangement, with corresponding graph $G(\mathcal{A}_n)$ defined as: vertex set $\mathbb{Z}/q\mathbb{Z} = \{0, 1, \dots, q-1\}$ and edges ij iff $i \equiv a_k \cdot j + b_k \pmod{q}$ for some $1 \leq i \neq j \leq n$, $1 \leq k \leq m$. Then if q is a large enough prime number, the number of n -element independent set is $\chi_{\mathcal{A}}(q)/n!$.*

The proposition is clear by the finite field method, since $\mathcal{B}_n \subset \mathcal{A}_n$, $(x_1, \dots, x_n) \in \mathbb{F}_q^n$ outside $\cup_{H \in \mathcal{A}_n} H$ must have distinct coordinates, while also in $G(\mathcal{A}_n)$, i and j are adjacent iff $i \equiv a_k \cdot j + b_k \pmod{q}$, which is exactly the defining equation of some hyperplane in \mathcal{A}_n (considered in \mathbb{F}_q). Since the independent set is an unordered set, we divide $\chi_{\mathcal{A}}(q)$ by $n!$.

Apply this proposition to Catalan arrangement \mathcal{C}_n , we know that the corresponding graph is the cycle graph C_q with q vertices, and hence the number of n -element independent set of C_q is $\chi_{\mathcal{C}_n}(q)/n!$ for q being a large enough prime number. In fact, this result is true for arbitrary $q \geq n \geq 1$, which can be obtained through combinatorial reasoning.

Many graphs can be constructed from hyperplane arrangements similarly to that in Proposition 1.4, whose characteristic polynomials hold combinatorial properties related to these graphs.

The main objective of this paper is to apply theories of rational hyperplane arrangements to solve the problem of counting independent sets in a disjoint union of the corresponding graphs, and explore invariants of such disjoint unions. Our main results are as follows.

For $a = \{a_1, \dots, a_m\}$ be a set of m multiplicative independent positive integers, i.e., if $a_1^{i_1} \cdots a_m^{i_m} = 1$ for some integers i_1, \dots, i_m , then $i_1 = \dots = i_m = 0$. Note this definition extends to positive rational numbers. We assume $a_i \neq 1$ for convenience. Then we define the following rational hyperplane arrangement as:

$$(1) \quad \mathcal{A}_n = \mathcal{B}_n \cup \{x_i = 0, \forall i, x_i = a_1 x_j, \forall i \neq j, \dots, x_i = a_m x_j, \forall i \neq j\},$$

Let $G(a, k)$ be the corresponding graph: vertex set $\{1, 2, \dots, k\}$ and edges ij if $i \equiv a_r j \pmod{k+1}$ for some r .

Under such construction, we first prove that when a_i 's are multiplicatively independent, the characteristic polynomial $\chi_{\mathcal{A}_n}(t)$ will remain the same for a fixed m , and secondly, we prove that for any disjoint union $G = G(a, k_1) + G(a, k_2) + \dots + G(a, k_s)$ of s such graphs, the number of n -element independent sets of G depends only on n, m and $\sum_i k_i$, and we achieve a closed-form solution based on the n to 1 covering from points in finite fields which characteristic polynomials are counting to the independent sets.

The rest of this paper is organized as follows. In Section 2, we use generating functions to prove the first main result, Theorem 2.1, which illustrates an important invariant of the characteristic polynomial of (1) and some combinatorial properties of the characteristic polynomial. As examples, we compute the characteristic polynomials through a computer algorithm in the Appendix A. In Section 3, we present proofs to our second main result, Theorem 3.1 and 3.4, and some generalizations. Examples are provided in Appendix B. In Section 4, we present a description of topological relationship between these arrangements and Catalan arrangements. Finally, we conclude with several open questions in Section 5.

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2. GENERATING FUNCTIONS AND CHARACTERISTIC POLYNOMIALS

We study the dependence of the characteristic polynomial of arrangement (1) on $a = \{a_1, a_2, \dots, a_m\} \subset \mathbb{P}$, where \mathbb{P} is the set of positive integers.

Theorem 2.1. *Let \mathcal{A}_n be the arrangement in \mathbb{R}^n with hyperplanes: $x_i = 0, \forall i, x_i = x_j, \forall i < j, x_i = a_1 x_j, \forall i \neq j, \dots, x_i = a_m x_j, \forall i \neq j$ where $\{a_1, \dots, a_m\} \subseteq \mathbb{P} - \{1\}$. Then for fixed m and n , the characteristic polynomial $\chi_{\mathcal{A}_n}$ is independent of the a_i 's, as long as the a_i 's are multiplicatively independent.*

Proof. Let $\mathcal{A}'_n = \mathcal{A}_n - \{x_i = 0, \forall i\}$. Repeated use of the deletion-restriction method in [2, Lemma 2.2] gives:

$$(2) \quad \chi_{\mathcal{A}'_n}(t) = \chi_{\mathcal{A}_n}(t) + n\chi_{\mathcal{A}_{n-1}}(t).$$

From this recursive formula, it suffices to show that $\chi_{\mathcal{A}'_n}$ is independent of the a_i 's, as long as they are multiplicative independent and m and n are fixed.

By Whitney's theorem, we have

$$\chi_{\mathcal{A}'_n}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}'_n} (-1)^{\#\mathcal{B}} t^{n - \text{rank}(\mathcal{B})}.$$

We dropped the condition for \mathcal{B} central since \mathcal{A}'_n is already central.

For each subarrangement $\mathcal{B} \subset \mathcal{A}'_n$, we associate to it a generalized graph $G_{\mathcal{B}}$ with vertex set $[n]$. For $x_i = x_j \in \mathcal{B}$ for some $i \neq j$, we connect i and j by an edge and label it by weight 1. For $x_j = a_l x_i \in \mathcal{B}$ for some $i \neq j$ and $a_l \neq 1$, we connect i and j by an arrow from i to j and label the weight a_l on the arrow. Hence there are $2m + 1$ possible types of edges or arrows between i and j ($i \neq j$), see Figure 3.

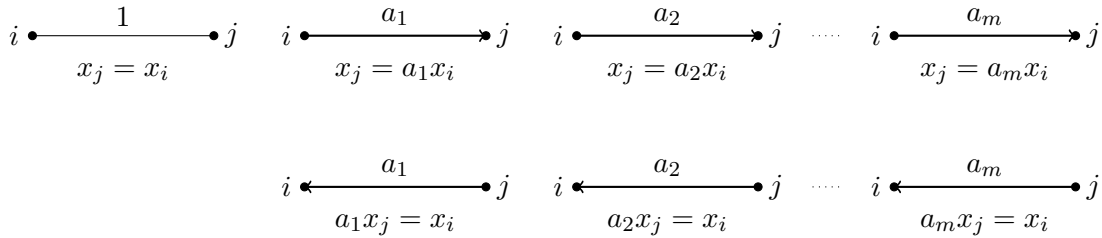


FIGURE 3. Different types of edges between vertices in graph $G_{\mathcal{B}}$.

Let $\Pi_{\mathcal{B}} \in \Pi_n$ be the partition corresponding to \mathcal{B} , where each block contains the vertices of the connected component of $G_{\mathcal{B}}$.

By exponential formula [7, §5.1], we have

$$(3) \quad \sum_{n \geq 0} \chi_{\mathcal{A}'_n}(t) \frac{x^n}{n!} = \exp \left[\sum_{n \geq 1} f_n(t) \frac{x^n}{n!} \right]$$

where

$$f_n(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}'_n, \Pi(\mathcal{B})=[n]} (-1)^{\#\mathcal{B}} t^{n - \text{rank}(\mathcal{B})}.$$

$\mathcal{B} \subseteq \mathcal{A}'_n, \Pi_{\mathcal{B}} = [n] \iff G_{\mathcal{B}}$ is connected $\iff \dim \cap_{H \in \mathcal{B}} H \leq 1 \iff \text{rank } \mathcal{B} \geq n - 1$ by the rank-nullity theorem.

$\chi_{\mathcal{A}'_0}(1) = \chi_{\mathcal{A}'_1}(1) = 1$ because $\hat{0} = \mathbb{R}^n$ is the only element in the intersection poset. By Theorem 1.2, $\chi_{\mathcal{A}'_n}(1) = 0$ for $i > 1$ since there are no relative bounded regions in a central arrangement. Plugging $t = 1$ into equation (3), we have

$$f_n(1) = (-1)^{n-1}(n-1)!.$$

Hence,

$$(4) \quad f_n(t) = b_n(t-1) + (-1)^{n-1}(n-1)!,$$

where

$$b_n = \sum_{\mathcal{B} \subset \mathcal{A}'_n, \text{rank}(\mathcal{B})=n-1} (-1)^{\#\mathcal{B}} = \sum_{\mathcal{B} \subset \mathcal{A}'_n, \text{rank}(\mathcal{B})=n-1} (-1)^{\#E(G_{\mathcal{B}})}$$

where $E(G_{\mathcal{B}})$ denotes the edges and arrows in $G_{\mathcal{B}}$.

For $\mathcal{B} \subset \mathcal{A}'_n$, the condition $\text{rank}(\mathcal{B}) = n-1$ is equivalent to saying in each cycle of $G_{\mathcal{B}}$, the arrows of weight a_i must appear the same number of times forward and reversed when traveling in one direction, since a_i 's are multiplicative independent. Hence we are counting the signed number $((-1)^{\#E(G_{\mathcal{B}})})$ of connected generalized graphs on $[n]$ with $m+1$ weights (or $2m+1$ types of edges and arrows between each pair of vertices as described above) with the described cycle condition. This value has nothing to do with exact values of a_i 's, as long as they are multiplicative independent. Hence we arrived at the conclusion.

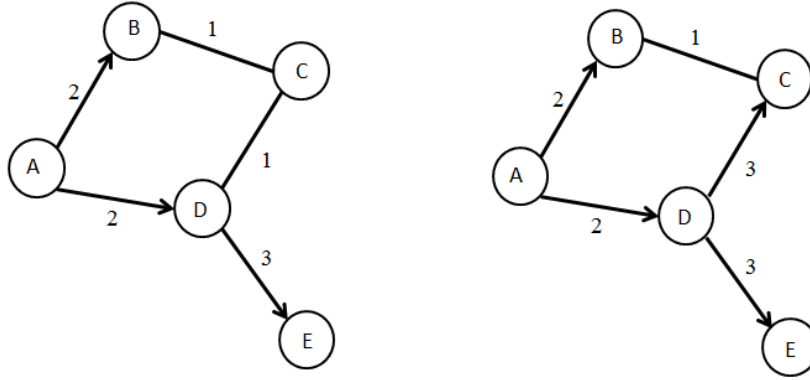


FIGURE 4. Two example components showing a valid (left) and invalid (right) set of edges, respectively.

□

Equations (2), (3) and (4) give

$$(5) \quad \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \exp \left[\sum_{n=1}^{\infty} \tilde{\chi}_{\mathcal{A}_n}(t) \frac{x^n}{n!} \right],$$

where $\tilde{\chi}_{\mathcal{A}_n}(t) = b_n(t-1)$.

Indeed the above reasoning still holds when a_i 's are positive rational numbers. I.e., equations (2)-(5) holds when $a_i \in \mathbb{Q}_+ - \{1\}$ and b_n is independent of a_i 's as long as they are multiplicative independent for fixed m and n .

Theorem 2.2. *Let \mathcal{A}_n be the arrangement in \mathbb{R}^n with hyperplanes: $x_i = 0$, $\forall i$, $x_i = x_j$, $\forall i < j$, $x_i = a_1 x_j$, $\forall i \neq j$, ..., $x_i = a_m x_j$, $\forall i \neq j$ where $\{a_1, \dots, a_m\} \subseteq \mathbb{Q}_+ - \{1\}$. Then,*

$$\sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} (-1)^n r(\mathcal{A}_n) \frac{x^n}{n!} \right)^{-\frac{t-1}{2}}.$$

Moreover, for fixed m and n , $r(\mathcal{A}_n)$ is independent of a_i 's as long as they are multiplicative independent.

Proof. Plugging $t = -1$ into equation (5) and applying Theorem 1.2, we have

$$\sum_{n=0}^{\infty} (-1)^n r(\mathcal{A}_n) \frac{x^n}{n!} = \exp \left[(-2) \sum_{n=1}^{\infty} b_n \frac{x^n}{n!} \right].$$

Combining (5) and the above equation gives the desired result. \square

3. MAIN THEOREMS IN COUNTING INDEPENDENT SETS OF GRAPHS

In Section 1, we mentioned in Proposition 1.4 that the number of n -element independent sets of a cycle graph of q vertices is $\frac{\chi_{C_n}(p)}{n!}$. Note that $\chi_{C_0}(0) = 1$ and $\chi_{C_{n \geq 1}}(0) = 0$. Now let $G = C_{k_1} + C_{k_2}$ be the disjoint union of two cycles C_{k_1} and C_{k_2} . The number of n -element independent sets of G is:

$$s_n = \sum_{j=0}^n \frac{\chi_{C_{n-j}}(k_1)}{(n-j)!} \cdot \frac{\chi_{C_j}(k_2)}{j!}$$

and

$$I_G(x) = \sum_{n=0}^{\infty} s_n x^n = I_{C_{k_1}}(x) I_{C_{k_2}}(x).$$

By exponential formula,

$$\sum_{n=0}^{\infty} \chi_{C_n}(k_i) \frac{x^n}{n!} = \exp \left[\sum_{n=1}^{\infty} \tilde{\chi}_{C_n}(k_i) \frac{x^n}{n!} \right], \quad i = 1, 2$$

where $\tilde{\chi}_{C_n}(t) = c_n t$ since Catalan arrangements form an exponential sequence of arrangements, see [2, pp.68].

Hence,

$$I_G(x) = \exp \left[\sum_{n=1}^{\infty} (\tilde{\chi}_{C_n}(k_1) + \tilde{\chi}_{C_n}(k_2)) \frac{x^n}{n!} \right] = \exp \left[\sum_{n=1}^{\infty} c_n (k_1 + k_2) \frac{x^n}{n!} \right] = \sum_{n=0}^{\infty} \chi_{C_n}(k_1 + k_2) \frac{x^n}{n!}.$$

Hence

$$s_n = \frac{\chi_{C_n}(k_1 + k_2)}{n!} = \frac{k_1 + k_2}{n!} \cdot (k_1 + k_2 - n - 1)_{n-1},$$

which shows the number of independent sets, s_n , only depends on n and $k_1 + k_2$. In other words, the number of independent sets of a disjoint union $G = C_{k_1} + C_{k_2}$ is equal to the number of independent sets of a single cycle graph $C_{k_1+k_2}$. This result remains true for $G = C_{k_1} + \dots + C_{k_s}$.

Things gets more complicated when $x_i = 0$, $\forall i$ is contained in the arrangement, which no longer forms an exponential sequence. We now study the disjoint union of graphs (maybe not connected) which correspond to the rational hyperplane arrangement (1).

Theorem 3.1. *Let $a = \{a_1, a_2, \dots, a_m\} \subset \mathbb{P} - \{1\}$ be a set of multiplicatively independent integers. For an integer $k \gg 1$, let $G(a, k)$ be the graph with vertex set $\{1, 2, \dots, k\}$ and edges ij if $i \equiv a_r j \pmod{k+1}$ for some r . Then the number of n -element independent sets of $G = G(a, k_1) + G(a, k_2) + \dots + G(a, k_s)$, where $k_1 + 1, k_2 + 1, \dots, k_s + 1$ are sufficiently large primes, depends only on n, m , and $\sum_i k_i$.*

Proof. The corresponding hyperplane arrangement in \mathbb{R}^n to the graph $G(a, k)$ is

$$\mathcal{A}_n = \mathcal{B}_n \cup \{x_i = 0, \forall i, x_i = a_1 x_j, \forall i \neq j, \dots, x_i = a_m x_j, \forall i \neq j\}.$$

By Proposition 1.4, $\chi_n(k+1)/n!$ is the number of n -element independent sets of the graph with vertex set $\{1, 2, \dots, k\}$ and edges ij if $i \equiv a_r j \pmod{k+1}$ for some r , as $(k+1)$ is a large prime

number. The number of n -element independent sets and the independence polynomial of G can be calculated as:

$$s_n = \sum_{j_1, j_2, \dots, j_s \geq 0, j_1 + j_2 + \dots + j_s = n} \frac{\chi_{\mathcal{A}_{j_1}}(k_1 + 1)}{j_1!} \cdot \frac{\chi_{\mathcal{A}_{j_2}}(k_2 + 1)}{j_2!} \dots \frac{\chi_{\mathcal{A}_{j_s}}(k_s + 1)}{j_s!}$$

$$I(x) = \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_1 + 1) \frac{x^n}{n!} \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_2 + 1) \frac{x^n}{n!} \dots \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_s + 1) \frac{x^n}{n!}.$$

Applying equation (5), $\tilde{\chi}_{\mathcal{A}_n}(k_i + 1) = b_n k_i$, and we immediately get

$$I(x) = \exp \left[\sum_{n=1}^{\infty} (\tilde{\chi}_{\mathcal{A}_n}(k_1 + 1) + \tilde{\chi}_{\mathcal{A}_n}(k_2 + 1) + \dots + \tilde{\chi}_{\mathcal{A}_n}(k_s + 1)) \frac{x^n}{n!} \right]$$

$$= \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_1 + k_2 + \dots + k_s + 1) \frac{x^n}{n!},$$

which implies that $s_n = \chi_{\mathcal{A}_n}(k_1 + k_2 + \dots + k_s + 1)/n!$.

The conclusion follows by applying Theorem 2.1. \square

Theorem 3.1 also implies that when considering the number of independent sets, a graph $G(a, p-1)$ can be viewed as a disjoint union $G(a, p-1) = G(a, p_1-1) + G(a, p_2-1)$ if p, p_1, p_2 are sufficiently large prime numbers with $p+1 = p_1 + p_2$. This condition is a weak form of Goldbach's Conjecture.

An generalized form of Theorem 3.1 is given in the following corollary:

Corollary 3.2. *Let $a = \{a_1, a_2, \dots, a_m\}$ and $b = \{b_1, b_2, \dots, b_m\}$ be two sets of positive integers where $a_i \neq b_i$ for all i . For an integer $k \gg 1$, let $G(a, b, k)$ be the graph with vertex set $\{1, 2, \dots, k\}$ and edges ij if $a_r i \equiv b_r j \pmod{(k+1)}$ for some r . Let G be any finite disjoint union $G(a, b, k_1) + G(a, b, k_2) + \dots + G(a, b, k_s)$ where $k_1 + 1, k_2 + 1, \dots, k_s + 1$ are sufficiently large primes, then the number of n -element independent sets of G depends on $n, a, b, \sum_i k_i$. Furthermore, if $\frac{a_i}{b_i}$'s are multiplicative independent, the number of n -element independent sets of G depends only on $n, m, \sum_i k_i$.*

Proof. This is immediate from Theorem 3.1 and Theorem 2.2. \square

Theorem 3.1 has an important applications when considering the deformation of the Braid arrangement, $x_i - x_j = a_1, a_2, \dots, a_m$, which has relatively bounded regions.

Lemma 3.3. *Let $a = \{a_1, a_2, \dots, a_m\}$ be a set of positive integers. For an integer $k \gg 1$, let $F(a, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r . Then the number of n -element independent sets of $F(a, k)$ is a degree n polynomial in k .*

Proof. Let $a_{\max} = \max\{a_1, a_2, \dots, a_m\}$. For all possible selections of n vertices in $\mathbb{Z}/k\mathbb{Z} = \{0, \dots, k\}$, we regard them as a disjoint union of clusters $D_1 \cup \dots \cup D_l$:

$\forall x_i \in D_i$ and $\forall x_j \in D_j$ when $\forall i \neq j$, $\min\{x_i - x_j \pmod{k}, x_j - x_i \pmod{k}\} > a_{\max}$;

$\forall x_i \in D_i$, $\exists x'_i \in D_i$ such that $x_i - x'_i \pmod{k}$ or $x'_i - x_i \pmod{k} \leq a_{\max}$.

Each element (vertex) is a cluster happens iff every vertex is separated from all other vertices in this n -element set by a minimum distance larger than $a_{\max} \pmod{k}$. Therefore the number of ways of choosing an n -element independent set which is the union of n clusters is $\chi_{\widehat{\mathcal{C}}_n}(t)/n!$. Here $\widehat{\mathcal{C}}_n$ is the arrangement in \mathbb{R}^n with hyperplanes $x_i - x_j = 0, 1, \dots, a_{\max}$. By Theorem 5.1 in [6], we know that

$$\chi_{\widehat{\mathcal{C}}_n}(t) = t(t - na_{\max} - 1)_{n-1}.$$

Hence the number of independent sets in this case is $k(k - na_{\max} - 1)_{n-1}/n!$, which is a polynomial in k with degree n .

For the case with $n-1$ clusters of vertices, the total number of independent sets is $b_1 k^{\binom{k-(n-1)a_{\max}-1}{n-2}}$ (see [3, Exercise 1.47]), which is a polynomial in k of degree $n-1$ and b_1 is a function on a , independent of k . Similarly, for other cases with $< n$ clusters of vertices, the total number of independent sets is a polynomial in k with degree less than n .

We arrive at the conclusion by adding all cases together. \square

Theorem 3.4. *Let $a = \{a_1, a_2, \dots, a_m\}$ be a set of positive integers. For an integer $k \gg 1$, let $F(a, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r . For some $k_1, k_2, \dots, k_s \gg 1$, let F be the disjoint union $F(a, k_1) + F(a, k_2) + \dots + F(a, k_s)$. Then the number of n -element independent sets of F depends only on a, n , and $\sum_i k_i$.*

Proof. The corresponding hyperplane arrangement in \mathbb{R}^n to the graph $F(a, k)$ is

$$\mathcal{A}_n = \mathcal{B}_n \cup \{x_i - x_j = a_1, a_2, \dots, a_m \mid 1 \leq i \neq j \leq n\}.$$

By Proposition 1.4, $\chi_{\mathcal{A}_n}(k)/n!$ is the number of n -element independent sets in $F(a, k)$ when p is a large prime number. Lemma 3.3 shows that the number of n -element independent sets of $F(a, k)$ is a polynomial of k with degree n , so it is $\chi_{\mathcal{A}_n}(k)/n!$.

Since \mathcal{A} forms an exponential sequence [2, Lecture 5], we have

$$\sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_i) \frac{x^n}{n!} = \exp \left[\sum_{n \geq 1} \tilde{\chi}_{\mathcal{A}_n}(k_i) \frac{x^n}{n!} \right], \quad i = 1, 2, \dots, s$$

where $\tilde{\chi}_{\mathcal{A}_n}(k_i) = b_n k_i$, $i = 1, 2, \dots, s$. Here, b_n only depends on n and a .

The independence polynomial of F can be calculated as:

$$\begin{aligned} I(x) &= \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} \sum_{\sum j_i = n, j_i \geq 0} \chi_{\mathcal{A}_{j_1}}(k_1) \chi_{\mathcal{A}_{j_2}}(k_2) \cdots \chi_{\mathcal{A}_{j_s}}(k_s) \frac{x^n}{j_1! j_2! \cdots j_s!} \\ &= \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_1) \frac{x^n}{n!} \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_2) \frac{x^n}{n!} \cdots \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_s) \frac{x^n}{n!} \\ &= \exp \left[\sum_{n \geq 1} (\tilde{\chi}_{\mathcal{A}_n}(k_1) + \tilde{\chi}_{\mathcal{A}_n}(k_2) + \cdots + \tilde{\chi}_{\mathcal{A}_n}(k_s)) \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n} \left(\sum_i k_i \right) \frac{x^n}{n!}. \end{aligned}$$

Hence $s_n = \frac{\chi_{\mathcal{A}_n}(\sum_i k_i)}{n!}$, and the conclusion follows. \square

Our proofs naturally lead us to the study of a broader family of graphs, namely those described by hyperplane arrangements without bounded regions.

Corollary 3.5. *Let $a = \{a_1, a_2, \dots, a_m\}$ and $b = \{b_1, b_2, \dots, b_m\}$ be two sets of positive integers where $a_i \neq b_i$ for all i . For an integer $k \gg 1$, let $F(a, b, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - a_r j \equiv b_r \pmod{k}$ for some r . For some $k_1, k_2, \dots, k_s \gg 1$, let F be the finite disjoint union $F(a, b, k_1) + F(a, b, k_2) + \dots + F(a, b, k_s)$. Then the number of n -element independent sets of F depends only on a, b, n, s , and $\sum_i k_i$. Furthermore, it is independent of s if and only if the corresponding hyperplane arrangement*

$$\mathcal{A}_n = \mathcal{B}_n \cup \{x_i - a_1 x_j = b_1, \forall i \neq j, x_i - a_2 x_j = b_2, \forall i \neq j, \dots, x_i - a_m x_j = b_m, \forall i \neq j\}$$

is not essential, i.e. $\text{rank}(\mathcal{A}_n) \leq n-1$.

Proof. Proposition 1.4 shows that $\chi_{\mathcal{A}_n}(k)/n!$ is the number of n -element independent sets of $F(a, b, k)$.

Similar reasoning gives

$$\sum_{n=0}^{\infty} \chi_{\mathcal{A}_n}(k_i) \frac{x^n}{n!} = \exp \left[\sum_{n \geq 1} \tilde{\chi}_{\mathcal{A}_n}(k_i) \frac{x^n}{n!} \right], \quad i = 1, 2, \dots, s$$

where $\tilde{\chi}_{\mathcal{A}_n}(k_i) = d_n k_i + c_n$, $i = 1, 2, \dots, s$, and d_n and c_n are only dependent on n and a . Denote by s_n the number of n -element independent sets of F . The independence polynomial of F can be expressed as:

$$\begin{aligned} I(x) &= \sum_{n=0}^{\infty} s_n x^n = \exp \left[\sum_{n \geq 1} (\tilde{\chi}_{\mathcal{A}_n}(k_1) + \tilde{\chi}_{\mathcal{A}_n}(k_2) + \dots + \tilde{\chi}_{\mathcal{A}_n}(k_s)) \frac{x^n}{n!} \right] \\ &= \sum_{n=0}^{\infty} \chi_{\mathcal{A}_n} \left(\sum_i k_i + (s-1) \cdot \frac{c_n}{d_n} \right) \frac{x^n}{n!}. \end{aligned}$$

Therefore, s_n of G only depends on $n, s, a, b, \sum_i k_i$. Its independent of s if and only if $c_n = 0$. Plugging $k_i = 0$, we get $\chi_{\mathcal{A}_{n \geq 1}}(0) = 0$. This is equivalent to saying $\text{rank}(\mathcal{A}_n) \leq n-1$, or \mathcal{A}_n is not essential. \square

We end this section by considering the graph with an additional vertex attaches via an edge to each vertex of the previous graph, as shown below.

Corollary 3.6. *Let $a = \{a_1, a_2, \dots, a_m\}$ be a set of positive integers. For an integer $k \gg 1$, let $F(a, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r , and let $\bar{F}(a, k)$ be the graph $F(a, k)$ with each vertex connected to a separated single vertex via an edge. For some $k_1, k_2, \dots, k_s \gg 1$, let \bar{F} be the disjoint union $\bar{F}(a, k_1) + \bar{F}(a, k_2) + \dots + \bar{F}(a, k_s)$. Then the number of n -element independent sets of G depends only on a, n , and $\sum_i k_i$.*

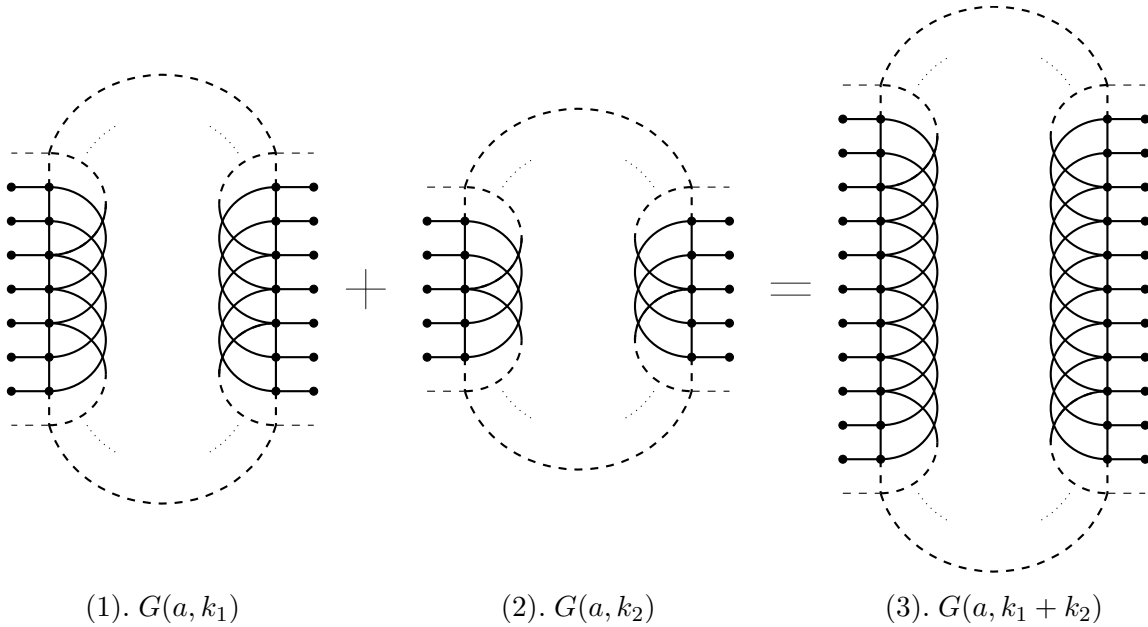


FIGURE 5. $F(a, k)$ with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r , and $\bar{F}(a, k)$ is formed by adding one vertex connecting to each vertex of $F(a, k)$.

Proof. We first prove the case $s = 2$, as shown in Figure 5. $F(a, k)$ corresponds to

$$\mathcal{A}_n = \mathcal{B}_n \cup \{x_i - x_j = a_1, a_2, \dots, a_m \mid 1 \leq i \neq j \leq n\}.$$

Then $\chi_{\mathcal{A}_n}(k)/n!$ is the number of n -element independent sets of $F(a, k)$. Hence the number of n -element independent sets of $\overline{F}(a, p)$ is:

$$s_n(\overline{F}(a, p)) = \sum_{i, j \geq 0, i+j=n} \frac{\chi_{\mathcal{A}_i}(p)}{i!} \cdot \frac{\chi_{\mathcal{B}_j}(p-i)}{j!}$$

where \mathcal{B}_n is the Braid arrangement.

Note that both \mathcal{A}_n and \mathcal{B}_n form exponential sequences of arrangements. We have the following calculation:

$$\begin{aligned} s_n(\overline{F}(a, k_1) + \overline{F}(a, k_2)) &= \sum_{i_1, i_2, j_1, j_2 \geq 0, i_1+i_2+j_1+j_2=n} \frac{\chi_{\mathcal{A}_{i_1}}(k_1)}{i_1!} \cdot \frac{\chi_{\mathcal{B}_{j_1}}(k_1-i_1)}{j_1!} \cdot \frac{\chi_{\mathcal{A}_{i_2}}(k_2)}{i_2!} \cdot \frac{\chi_{\mathcal{B}_{j_2}}(k_2-i_2)}{j_2!} \\ &= \sum_{i_1, i_2, j \geq 0, i_1+i_2+j=n} \left[\frac{\chi_{\mathcal{A}_{i_1}}(k_1)}{i_1!} \cdot \frac{\chi_{\mathcal{A}_{i_2}}(k_2)}{i_2!} \right] \cdot \frac{\chi_{\mathcal{B}_j}(k_1+k_2-i_1-i_2)}{j!} \\ &= \sum_{i, j \geq 0, i+j=n} \frac{\chi_{\mathcal{A}_i}(k_1+k_2)}{i!} \cdot \frac{\chi_{\mathcal{B}_j}(k_1+k_2-i)}{j!} \\ &= s_n(\overline{F}(a, k_1+k_2)). \end{aligned}$$

This implies the number of independent sets of the disjoint union $\overline{F} = \overline{F}(a, k_1) + \overline{F}(a, k_2)$ only depends on a , n and $k_1 + k_2$. By induction, the result remains true for finite disjoint unions $\overline{F} = \overline{F}(a, k_1) + \overline{F}(a, k_2) + \dots + \overline{F}(a, k_s)$. \square

We provide few examples in Appendix B.

4. RELATIONS WITH CATALAN ARRANGEMENTS

In the previous section, we found that the problem of counting independent sets of disjoint union of certain types of graphs have solutions in the form of characteristic polynomials of rational hyperplane arrangements. This reduces to the study of computing characteristic polynomials of certain graph related arrangements. Deformations of the Braid arrangement, such as the Catalan arrangement, semiorder arrangement and the Shi arrangement, have been extensively studied by multiple approaches [8]. Here, we regard (1) as a deformation of the Braid arrangement in order to find an alternative way to calculate characteristic polynomials by topologically isomorphic studies.

Theorem 4.1. *Let \mathcal{A}_n be the arrangement in \mathbb{R}^n with hyperplanes:*

$$\begin{array}{ll} x_i = 0 & \forall i \\ x_i = x_j & \forall i < j \\ x_i = a_1 x_j & \forall i \neq j \\ \dots & \\ x_i = a_m x_j & \forall i \neq j \end{array}$$

where $\{a_1, \dots, a_m\} \subseteq \mathbb{P} - \{1\}$, and $\widetilde{\mathcal{C}}_n$ be an extended Catalan arrangement in \mathbb{R}^n with hyperplanes:

$$\begin{aligned} x_i - x_j &= 0 & \forall i < j \\ x_i - x_j &= 1 & \forall i \neq j \\ x_i - x_j &= \log a_2 / \log a_1 & \forall i \neq j \\ &\dots & \\ x_i - x_j &= \log a_m / \log a_1 & \forall i \neq j. \end{aligned}$$

Then $\chi_{\mathcal{A}_n}(t) = \chi_{\widetilde{\mathcal{C}}_n}(t-1)$.

Proof. First we consider the number of regions, $r(\mathcal{A}_n)$ in different regions separated by hyperplanes $\{x_i = 0, \forall i\}$.

In $\cap_{i=1}^n \{x_i > 0\}$, exchange variables via $x_i = a_1^{y_i}$, then the arrangement becomes $\widetilde{\mathcal{C}}_n$ which indicates the number of regions in $\cap_{i=1}^n \{x_i > 0\}$ is equal to $r(\widetilde{\mathcal{C}}_n)$.

In the big regions which are the intersection of i positive half-spaces and $n-i$ negative half-spaces, exchange variables via $x_i = a_1^{y_i}$ when $x_i > 0$ and $x_i = -a_1^{y_i}$ when $x_i < 0$, then the number of regions is equal to $r(\widetilde{\mathcal{C}}_i) \cdot r(\widetilde{\mathcal{C}}_{n-i})$. Hence

$$\begin{aligned} r(\mathcal{A}_n) &= \sum_{i=0}^n \binom{n}{i} r(\widetilde{\mathcal{C}}_i) r(\widetilde{\mathcal{C}}_{n-i}). \\ \sum_{n=0}^{\infty} r(\mathcal{A}_n) \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n}{i} r(\widetilde{\mathcal{C}}_i) r(\widetilde{\mathcal{C}}_{n-i}) \frac{x^n}{n!} = \left(\sum_{n=0}^{\infty} r(\widetilde{\mathcal{C}}_n) \frac{x^n}{n!} \right)^2. \end{aligned}$$

Using Theorem 5.17 in [2, pp.67] and Theorem 2.3, we have

$$\begin{aligned} \sum_{n \geq 0} \chi_{\widetilde{\mathcal{C}}_n}(t) \frac{(-x)^n}{n!} &= \left(\sum_{n \geq 0} r(\widetilde{\mathcal{C}}_n) \frac{x^n}{n!} \right)^{-t} \\ \sum_{n \geq 0} \chi_{\mathcal{A}_n}(t) \frac{(-x)^n}{n!} &= \left(\sum_{n \geq 0} r(\mathcal{A}_n) \frac{x^n}{n!} \right)^{-(t-1)/2}. \end{aligned}$$

The above three equations leads to $\chi_{\mathcal{A}_n}(t) = \chi_{\widetilde{\mathcal{C}}_n}(t-1)$. □

As an application, we give the following two results.

Proposition 4.2. *Let \mathcal{A}_n be the arrangement in \mathbb{R}^n with hyperplanes: $x_i = 0, \forall i, x_i = x_j, \forall i < j, x_i = a^1 x_j, \forall i \neq j, \dots, x_i = a^m x_j, \forall i \neq j$ for some $a \in \mathbb{P} - \{1\}$. Then*

$$\chi_{\mathcal{A}_n}(t) = (t-1) \prod_{j=1}^{n-1} (t-1-mn-j).$$

Proof. This is done directly by plugging $a_1 = a, a_2 = a^2, \dots, a_k = a^k$ in Theorem 4.1 and applying [6, Thm 5.1]. □

Proposition 4.3. *Let \mathcal{A}_n be the arrangement in \mathbb{R}^n with hyperplanes $x_i = 0, \forall i, x_i = x_j, \forall i < j, x_i = ax_j, \forall i < j$ for some $a \in \mathbb{P} - \{1\}$. Then*

$$\chi_{\mathcal{A}_n}(t) = (t-1)(t-1-n)^{n-1}.$$

Proof. Theorem 4.1 remains true if we replace $\forall i \neq j$ by $\forall i < j$.

Utilizing the characteristic polynomial of Shi arrangement defined by $x_i - x_j = 0, 1$ for $1 \leq i < j \leq n$ in [2, pp.64]: $\chi_{\mathcal{S}_n}(t) = t(t-n)^{n-1}$, we have: $\chi_{\mathcal{A}_n}(t) = (t-1)(t-1-n)^{n-1}$. □

In fact, when a_1, a_2, \dots, a_m are multiplicatively independent positive integers, $1, \log a_2 / \log a_1, \dots, \log a_m / \log a_1$ are independent over \mathbb{Q} , and hence $1, \log a_2 / \log a_1, \dots, \log a_m / \log a_1$ are generic over \mathbb{R} , i.e., when we write the hyperplanes as $H_i = \ker(L_i(v) = a_i)$, where L_i 's are not necessarily distinct linear forms and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, then

$$H_{i_1} \cap \dots \cap H_{i_k} \neq \emptyset \iff L_{i_1}, \dots, L_{i_k} \text{ linearly independent.}$$

Then by [2, pp.22]

$$\chi_{\widetilde{\mathcal{C}_n}}(t) = \sum_{\mathcal{B} \subset \widetilde{\mathcal{C}_n}} (-1)^{\#\mathcal{B}} t^{n - \#\mathcal{B}}$$

where \mathcal{B} ranges over linearly independent subsets of $\widetilde{\mathcal{C}_n}$. We say a set of hyperplanes linearly independent iff their normal vectors are linearly independent, hence has nothing to do with the exact values of a_1, \dots, a_m for fixed m . Therefore we obtained an alternative proof for Theorem 2.1.

5. CONCLUSION AND SOME OPEN PROBLEMS

In this paper, we computed characteristic polynomials of a class of rational hyperplane arrangements and numbers of independent sets in their corresponding graphs. In particular, we proved the theorem that for graphs $G(k)$ with k vertices corresponding to the rational hyperplane arrangements, the number of n -element independent sets of any disjoint union $G = G(k_1) + G(k_2) + \dots + G(k_s)$ depends only on the total number of vertices $\sum_i k_i$, instead of the size of each individual subgraph. Moreover, we generalized this result to a number of new results with a broader condition and discovered that an extended Catalan arrangement, $\widetilde{\mathcal{C}_n}$, is topologically embedded in \mathcal{A}_n , and proved that these characteristic polynomials differed by a shift: $\chi_{\mathcal{A}_n}(t) = \chi_{\widetilde{\mathcal{C}_n}}(t - 1)$.

One direction for future research is classifying various graphs and their corresponding hyperplane arrangements. With as many arrangements in the classification, we would be able to at least provide systematic solutions to a large group of independent set counting problems. Analysis and experiments by computer calculations suggest the following conjecture.

Conjecture 5.1. *Let $a = \{a_1, a_2, \dots, a_m\}$ be a set of positive integers. For an integer $k \gg 1$, let $F(a, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r , and let $\overline{F}(a, k)$ be the graph $F(a, k)$ with each vertex connected to a separated identical subgraph via an edge. For some $k_1, k_2, \dots, k_s \gg 1$, let \overline{F} be the finite disjoint union $\overline{F}(a, k_1) + \overline{F}(a, k_2) + \dots + \overline{F}(a, k_s)$. Then the number of n -element independent sets of \overline{F} depends only on a, n , and $\sum_i k_i$, instead of individual k_i 's or the number s of k_i 's.*

We have proved the case when each connected subgraph is single vertex. For arbitrary subgraphs connecting to the base graph, no proof has been achieved yet. Further observations suggest the following conjecture.

Conjecture 5.2. *Let $a = \{a_1, a_2, \dots, a_m\}$ and $b = \{b_1, b_2, \dots, b_m\}$ be two sets of positive integers. For an integer $k \gg 1$, let $F(a, b, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - a_r j \equiv b_r \pmod{k}$ for some r , and let $\overline{F}(a, b, k)$ be the graph $F(a, b, k)$ with each vertex sequentially connected to one vertex of a cycle graph C_k . For some $k_1, k_2, \dots, k_s \gg 1$, let \overline{F} be the finite disjoint union $\overline{F}(a, b, k_1) + \overline{F}(a, b, k_2) + \dots + \overline{F}(a, b, k_s)$. Then the number of n -element independent sets of \overline{F} depends only on a, b, n , and $\sum_i k_i$, iff corresponding hyperplane arrangement \mathcal{A}_n is not essential.*

Computer calculations also suggest that characteristic polynomial starts to deviate from exact number of k -element independent sets when the size of the corresponding graph is not large enough compared to k , but it is still a reasonably good approximation for many graphs as long as the degree of each vertex is less than k/n . So, there are a few of open questions: how can we quantify this approximation for a disjoint union, and how can the characteristic polynomial of rational hyperplane arrangement be used to generate a good approximation for the maximal independent set in a disjoint union.

APPENDIX A. COMPUTER CALCULATION OF CHARACTERISTIC POLYNOMIALS

We take a case study on (1) with $m = 2$: let $\mathcal{A}_n = \mathcal{B}_n \cup \{x_i = 0, \forall i, x_i = a_1 x_j, \forall i \neq j, x_i = a_2 x_j, \forall i \neq j\}$ where $a_1, a_2 \in \mathbb{P} - \{1\}$ and $a_1 \neq a_2$.

Instead of using brute-force algorithm based upon the definition of characteristic polynomial, we develop an algorithm based on Proposition 1.4 to calculate the number $s_3(q)$ of 3-element independent sets of the corresponding graph on $\{1, 2, \dots, q-1\}$, for various prime number q with different pairs of (a_1, a_2) . Calculated values of $\frac{3!}{q-1} \cdot s_3$ for varied (a_1, a_2) and q are listed in Table 1. This program has a better time complexity, finishing in a reasonable few seconds.

(a_1, a_2)	$q = 23$	$q = 29$	$q = 31$	$q = 37$	$q = 41$	$q = 47$	$q = 53$	$q = 59$	$q = 61$...	$q=199$
(2, 3)	216	426	512	818	1062	1196	1488	1986	2556	...	36296
(2, 5)	210	426	510	818	1062	1196	1488	1986	2556	...	36296
(3, 5)	216	426	510	812	1062	1196	1488	1986	2556	...	36296
(5, 7)	216	420	510	818	1062	1196	1488	1986	2556	...	36296
(2, 4)	210	420	500	812	1056	1190	1482	1980	2550	...	36290

TABLE 1. $\frac{3!}{q-1} \cdot s_3$, where s_3 is the calculated number of 3-element independent sets of the graph.

As the prime number q is large enough, the characteristic polynomial $\chi_{\mathcal{A}_3}(q) = 3!s_3$. We observe that with large enough q , $\frac{\chi_{\mathcal{A}_3}(q)}{q-1} = \frac{3!}{q-1} \cdot s_3$ is independent of choosing (a_1, a_2) as long as a_1, a_2 are multiplicatively independent. If a_1, a_2 are not multiplicatively independent, for example $(a_1, a_2) = (2, 4)$, the values (in red) are different. Because $\frac{\chi_{\mathcal{A}_3}(q)}{q-1}$ is a degree 2, we can calculate it by polynomial interpolation. Similarly, we find the characteristic polynomials for a multiplicatively independent pair (a_1, a_2) for small dimensions as below:

$$\begin{aligned}
\chi_{\mathcal{A}_2}(q) &= (q-1)(q-6) \\
\chi_{\mathcal{A}_3}(q) &= (q-1)(q^2 - 17q + 78) \\
\chi_{\mathcal{A}_4}(q) &= (q-1)(q^3 - 33q^2 + 386q - 1608) \\
\chi_{\mathcal{A}_5}(q) &= (q-1)(q^4 - 54q^3 + 1151q^2 - 11514q + 45840) \\
\chi_{\mathcal{A}_6}(q) &= (q-1)(q^5 - 80q^4 + 2675q^3 - 46840q^2 + 431004q - 1675440) \\
\chi_{\mathcal{A}_7}(q) &= (q-1)(q^6 - 111q^5 + 5335q^4 - 142365q^3 + 2230264q^2 - 19515684q + 74864160).
\end{aligned}$$

APPENDIX B. EXAMPLES OF THEOREM 3.1 AND THEOREM 3.4

Example B.1. Let $a = \{3, 5\}$ (3 and 5 are multiplicatively independent). Let $G(a, k)$ be the graph with vertex set $[k]$ and edges ij if $i \equiv 3j \pmod{k+1}$, or $i \equiv 5j \pmod{k+1}$. As shown in Figure 6, the number of n -element independent sets of the disjoint union $G = G(a, 18) + G(a, 22)$ is equal to that of the graph $G(a, 40)$, which can readily be calculated by a closed form formula, i.e., $s_n = \chi_{\mathcal{A}_n}(41)/n!$, which is independent of a .

Example B.2. Let $a = \{2, 3\}$. Let $G(a, k)$ be the graph with vertex set $[k]$ and edges ij if $i \equiv 2j \pmod{k+1}$, or $i \equiv 3j \pmod{k+1}$. In figure 7, we have $G(a, 18)$ and $G(a, 22)$, the latter of which has two components.

The number of independent sets of the disjoint union of $G(a, 18)$ and $G(a, 22)$ is equal to that of the graph $G(a, 40)$, as shown in Figure 7. Note that the number of independent sets of the disjoint union is independent of a , since 2 and 3 are multiplicative independent.

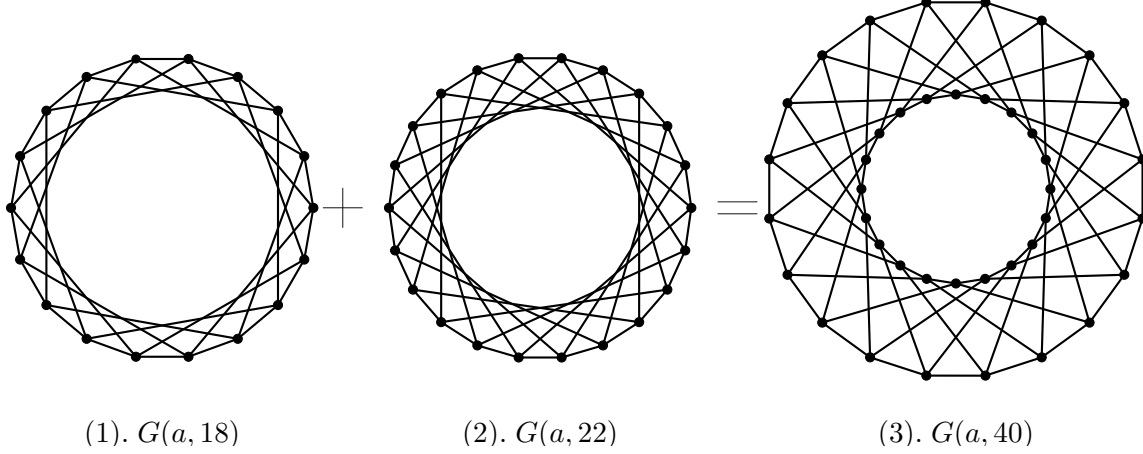


FIGURE 6. $G(a = \{3, 5\}, k)$ on $[k]$ with edges ij if $i \equiv a_r j \pmod{(k+1)}$ for some r .

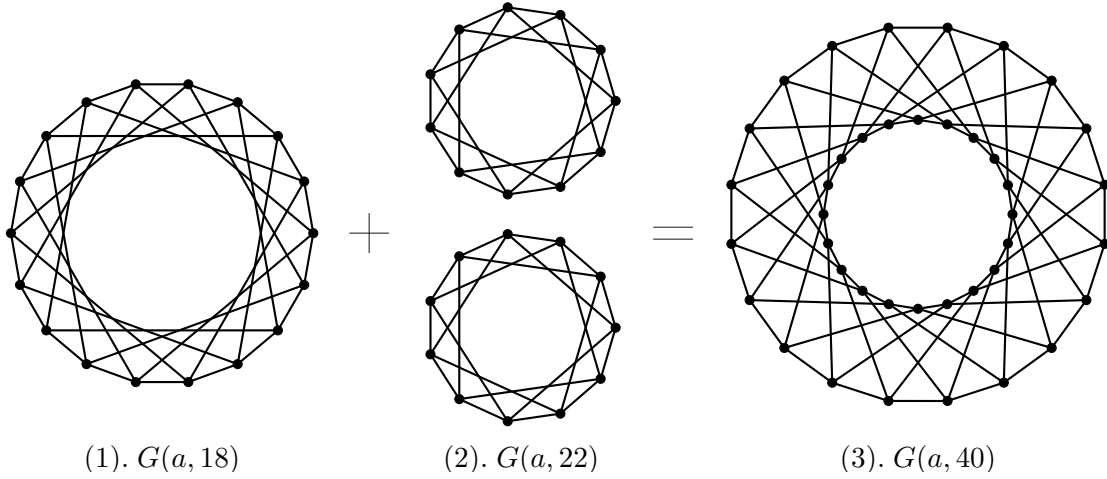


FIGURE 7. $G(a = \{2, 3\}, k)$ on $[k]$ with edges ij if $i \equiv a_r j \pmod{(k+1)}$ for some r .

Example B.3. Let $a = \{1, 3\}$. Let $F(a, k)$ be the graph with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv 1 \pmod{k}$, or $i - j \equiv 3 \pmod{k}$, as shown in Figure 8. Then the number of n -element independent sets of the disjoint union $F = F(a, k_1) + F(a, k_2)$ is equal to that of the graph $F(a, k_1 + k_2)$, which can readily be calculated by a closed form formula, i.e., $s_n = \chi_{\mathcal{A}_n}(k_1 + k_2)/n!$.

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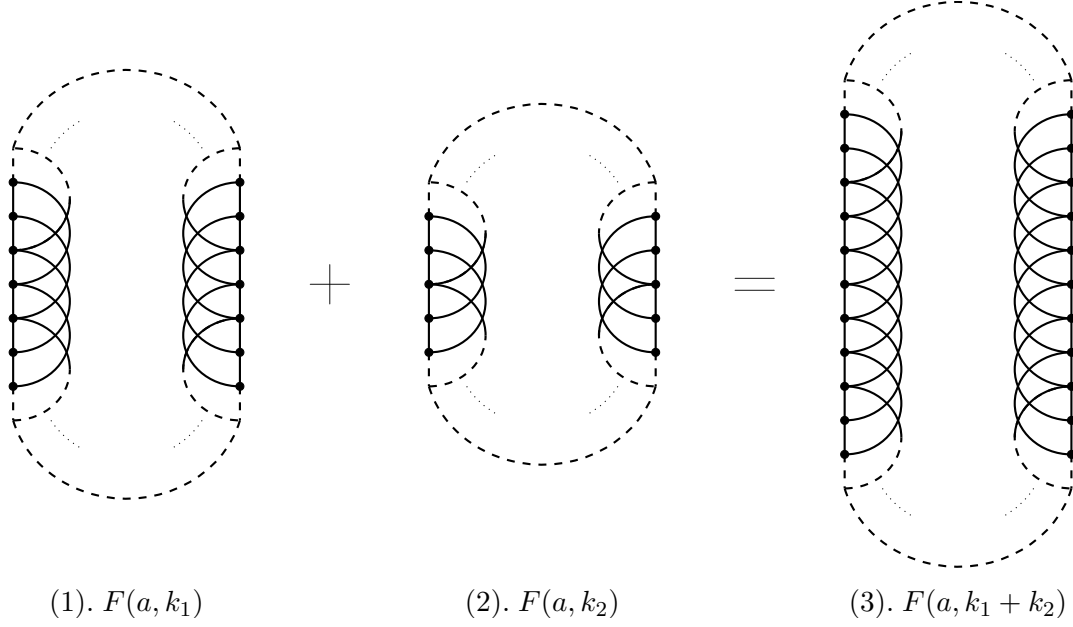


FIGURE 8. $F(a = \{1, 3\}, k)$ with vertex set $\mathbb{Z}/k\mathbb{Z}$ and edges ij if $i - j \equiv a_r \pmod{k}$ for some r .

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